

A note on equivariant eta forms

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Abstract

In this note, we prove the regularity of eta forms by the Clifford asymptotics. Then we generalize this result to the equivariant case.

Keywords: Eta forms; Equivariant eta forms; Clifford asymptotics

1 Introduction

In [APS], Atiyah, Patodi and Singer proved the Atiyah-Patodi-Singer index theorem for manifolds with boundary and they introduced the eta invariants. Bismut and Freed gave a simple proof of the regularity of eta invariants in [BF]. Bismut and Cheeger generalized the Atiyah-Patodi-Singer index theorem to the family case in [BC1,2]. They used the eta forms for families of Dirac operators. The regularity of eta forms was proved by the probability method in [BGS]. Donnelly generalized the Atiyah-Patodi-Singer index theorem to the equivariant case and introduced the equivariant eta invariants in [D]. Zhang proved the regularity of the equivariant eta invariants by the Clifford asymptotics in [Z2]. In this note, we firstly prove the regularity of eta forms by the Clifford asymptotics. Then we define the equivariant eta forms and prove their regularity.

2 The regularity of eta forms

Let M be a $n + m$ dimensional compact connected manifold and n be an odd integer and B be a m dimensional compact connected manifold. We assume that $\pi : M \rightarrow B$ is a submersion of M onto B , which defines a fibration of M with fibre G . For $y \in B$, $\pi^{-1}(y)$ is then a submanifolds G_y of M . TG denotes the n -dimensional vector bundle on M whose fibre $T_x G$ is the tangent space at x to the fibre $G_{\pi x}$. We assume that M and B are oriented. Taking the orthogonal bundle of TG in TM with respect to any Riemannian metric, determines a smooth horizontal subbundle $T^H M$, i.e. $TM = T^H M \oplus TG$. Vector fields $X \in TB$ will be identified with their horizontal lifts $X \in T^H M$, moreover $T_x^H M$ is isomorphic to $T_{\pi(x)} B$ via π_* . Recall that B is Riemannian, so we can lift the Euclidean scalar product g_B of TB to $T^H M$. And we assume that TG is endowed with a scalar product g_G . Thus we can introduce in TM a new scalar product $g_B \oplus g_G$, and denote by ∇^L the Levi-Civita connection on TM with respect to this metric. Let ∇^B denote the Levi-Civita connection on TB and

we still denote by ∇^B the pullback connection on $T^H M$. Let $\nabla^G = P_G(\nabla^L)$ where P_G denotes the projection to TG . Let $\nabla^\oplus = \nabla^B \oplus \nabla^G$ and $S = \nabla^L - \nabla^\oplus$ and T is the torsion tensor of ∇^\oplus . Let $SO(TG)$ be the $SO(n)$ bundle of oriented orthonormal frames in TG . Now we assume that bundle TG is spin. Let $S(TG)$ be the associated spinors bundle and ∇^G can be lifted to give a connection on $S(TG)$. Let D be the tangent Dirac operator. Let K be the scalar curvature of fiber G and $e_1(x), \dots, e_n(x)$ denote the orthonormal frame of TG . If $A(Y)$ is any 0 order operator depending linearly on $Y \in TM$, we define the operator $(\nabla_{e_i} + A(e_i))^2$ as follows

$$(\nabla_{e_i} + A(e_i))^2 = \sum_1^n (\nabla_{e_i(x)} + A(e_i(x)))^2 - \nabla_{\sum_j \nabla_{e_j} e_j} - A(\sum_j \nabla_{e_j} e_j). \quad (2.1)$$

Let Tr^{even} denote taking trace on the coefficients of even forms on B . Let $c(T) = \sum_{1 \leq \alpha < \beta \leq m} dy_\alpha dy_\beta c(T(\frac{\partial}{\partial y_\alpha}, \frac{\partial}{\partial y_\beta}))$, and

$$I^t = -t(\nabla_{e_i}^G + \frac{1}{2\sqrt{t}} \langle S(e_i)e_j, f_\alpha \rangle e_j dy_\alpha + \frac{1}{4t} \langle S(e_i)f_\alpha, f_\beta \rangle dy_\alpha dy_\beta)^2 + \frac{tK}{4}. \quad (2.2)$$

We make the following definition of the eta forms.

Definition 2.1 $\hat{\eta}$ denotes the even degree form on B

$$\hat{\eta} = \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} \text{Tr}^{\text{even}}[(D + \frac{c(T)}{4t})\exp(I^t)] dt/t^{\frac{1}{2}}. \quad (2.3)$$

In order to prove that (2.3) is well defined, we need to prove the regularity at origin and $+\infty$. Firstly we prove the regularity at $+\infty$. We consider the two terms $\text{Tr}^{\text{even}}[D\exp(I^t)]$ and $\text{Tr}^{\text{even}}[\frac{c(T)}{4t}\exp(I^t)]$. For the first term, considering D , we can assume that D is invertable and using the discussions as in [W, pp.148-150], then we can get the regularity at $+\infty$ of the first term. Since $\frac{c(T)}{4t}$ is a bounded operator, using the discussions in [BGV, p. 275], we can get the regularity at $+\infty$ of the second term. Nextly, we prove the regularity at origin. Fixing a $y \in B$ and considering the fibre G_y , let $\exp(I^t)(x, x')$ for $x, x' \in G_y$ denote the heat kernel of I^t (see [BGV]). By a simple discussion, we can get

$$\text{Tr}^{\text{even}}[(D + \frac{c(T)}{4t})\exp(I^t)] = \int_{G_y} \text{Tr}^{\text{even}}[(D + \frac{c(T)}{4t})\exp(I^t)(x, x)] dx. \quad (2.4)$$

Let $I^1 = I^t|_{t=1}$. By the Duhamel principle, we can get

$$\text{Tr}^{\text{even}}[\exp(t(I^1 + dt(D + \frac{c(T)}{4})))] = \text{Tr}^{\text{even}}[\exp(tI^1)] + tdt\text{Tr}^{\text{even}}[(D + \frac{c(T)}{4})\exp(tI^1)]. \quad (2.5)$$

Let ϕ be a smooth function defined locally in a neighborhood of x , denote the degree of zero of ϕ at x by $v(\phi)$, to every

$$\alpha(x') = \phi_{l_1}(x') \frac{\partial}{\partial x_{i_1}} \phi_{l_2}(x') \cdots \frac{\partial}{\partial x_{i_m}} \phi_{l_{m+1}}(x') dy_{\alpha_1} \cdots dy_{\alpha_p} dt$$

$$e_{j_1} \cdots e_{j_s} : S(TG)_{x'} \rightarrow S(TG)_{x'}; \alpha_i \neq \alpha_j \ (i \neq j); \ j_a \neq j_t \ (a \neq t), \quad (2.6a)$$

we define

$$\chi(\alpha) = m + p + s + 1 - v(\phi_1 \cdots \phi_{m+1}). \quad (2.6b)$$

We call $\alpha(x')$ an even (odd) element if $p + s$ is a even (odd) integer and we denote $\{\chi < m\}$ the linear space generated by all the elements α for which $\chi(\alpha) < m$ and denote $(\chi < m)$ an element of $\{\chi < m\}$, e.g. $\omega = \omega' + (\chi < m)$ means that there exists a $\beta \in \{\chi < m\}$ such that $\omega = \omega' + \beta$. Set

$$h(x) = 1 + \frac{1}{2} dt \sum_{i=1}^n x_i e_i. \quad (2.7)$$

Then we have

$$h e_i h^{-1} = e_i + dt(\chi \leq -1); \quad h(\frac{1}{2} e_i dt) h^{-1} = \frac{1}{2} e_i dt, \quad (2.8)$$

$$h \nabla_{e_i}^G h^{-1} = \nabla_{e_i}^G - \frac{1}{2} dt e_i + dt(\chi \leq -1); \quad (2.9)$$

$$h(\frac{1}{2} < S(e_i) e_j, f_\alpha > e_j dy_\alpha) h^{-1} = \frac{1}{2} < S(e_i) e_j, f_\alpha > e_j dy_\alpha + dt(\chi \leq -1); \quad (2.10)$$

$$h(\frac{1}{4} < S(e_i) f_\alpha, f_\beta > dy_\alpha dy_\beta) h^{-1} = \frac{1}{4} < S(e_i) f_\alpha, f_\beta > dy_\alpha dy_\beta. \quad (2.11)$$

By the proposition 2.10 in [BGS], we have

$$\begin{aligned} I^1 + dt(D + \frac{c(T)}{4}) &= -(\nabla_{e_i}^G + \frac{1}{2} < S(e_i) e_j, f_\alpha > e_j dy_\alpha \\ &+ \frac{1}{4} < S(e_i) f_\alpha, f_\beta > dy_\alpha dy_\beta - \frac{1}{2} e_i dt)^2 + \frac{K}{4}, \end{aligned} \quad (2.12)$$

By (2.7)-(2.12), we have

$$h[I^1 + dt(D + \frac{c(T)}{4})] h^{-1} = I^1 + dtu, \quad (2.13)$$

where $\chi(u) \leq 0$. For $t > 0$, by [Z1, (4.17)], we have

$$\exp(tI^1)(x, x') = \frac{e^{-d(x, x')^2/4t}}{(4\pi t)^{\frac{n}{2}}} \left(\sum_{i=0}^{[\frac{n}{2}] + [\frac{m}{2}] + 2} U_i t^i + o(t^{[\frac{n}{2}] + [\frac{m}{2}] + 2}) \right). \quad (2.14)$$

For $I^1 + dtu$, similar to (4.17) in [Z1], we have

$$[\hat{d} + x_i(b_i \Gamma_{ij}^\alpha e_j dy_\alpha + c_i \Gamma_{i\alpha}^\beta dy_\alpha dy_\beta + B_i) + r] \widetilde{U_r} + dt A \widetilde{U_r} = (I^1 + dtu) \widetilde{U_{r-1}} \quad (2.15)$$

where $\widetilde{U_r} = U_r + dtV_r$ and $\widetilde{U_{-1}} = 0$ and $\chi(A) \leq -2$ and see [Z1] for \hat{d} , $b_i\Gamma_{ij}^\alpha$, B_i . By (2.15), we have

$$\exp(t(I^1 + dtu))(x, x') = \frac{e^{-d(x, x')^2/4t}}{(4\pi t)^{\frac{n}{2}}} \left(\sum_{i=0}^{[\frac{n}{2}] + [\frac{m}{2}] + 2} (U_i + dtV_i)t^i + o(t^{[\frac{n}{2}] + [\frac{m}{2}] + 2}) \right), \quad (2.16)$$

where $\chi(U_i) \leq 2i$, $\chi(V_i) \leq 2(i-1)$ and U_i, V_i contain no dt . U_i is an even element and V_i is an odd element. By (2.13), we have

$$\exp[t(I^1 + dt(D + \frac{c(T)}{4}))](x, x) = h^{-1}(x)\exp[t(I^1 + dtu)]h(x). \quad (2.17)$$

By (2.5) and (2.17), we get

$$\begin{aligned} tdt\text{Tr}^{\text{even}}[(D + \frac{c(T)}{4})\exp(tI^1)(x, x)] &= \text{Tr}^{\text{even}}\{h^{-1}(x)\exp[t(I^1 + dtu)]h(x)\} \\ &\quad - \text{Tr}^{\text{even}}[\exp(tI^1)(x, x)]. \end{aligned} \quad (2.18)$$

By (2.8), (2.14) and (2.16), we get

$$\begin{aligned} &\text{Tr}^{\text{even}}\{h^{-1}(x)\exp[t(I^1 + dtu)]h(x)\} - \text{Tr}^{\text{even}}[\exp(tI^1)(x, x)] \\ &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \text{Tr}^{\text{even}} \left(\sum_{i=0}^{[\frac{n}{2}] + [\frac{m}{2}] + 2} dtW_i t^i + o(t^{[\frac{n}{2}] + [\frac{m}{2}] + 2}) \right), \end{aligned} \quad (2.19)$$

where $\chi(W_i) \leq 2(i-1)$ is an odd element. Let ϕ_t be the rescaling operator defined by $\phi_t(dy_\alpha) = \frac{1}{\sqrt{t}}dy_\alpha$, $\phi_t(dt) = \frac{1}{\sqrt{t}}dt$. Then by (2.18) and (2.19), we have

$$\text{Tr}^{\text{even}}[(D + \frac{c(T)}{4t})\exp(I^t)(x, x)] = \frac{1}{(4\pi t)^{\frac{n}{2}}t} \phi_t \left\{ \text{Tr}^{\text{even}} \left(\sum_{i=0}^{[\frac{n}{2}] + [\frac{m}{2}] + 2} W_i t^i + o(t^{[\frac{n}{2}] + [\frac{m}{2}] + 2}) \right) \right\}. \quad (2.20)$$

Lemma 2.2 Suppose $i \leq [\frac{n}{2}] + 1$. If W is an odd element and $\chi(W_i) \leq 2(i-1)$, then $\text{Tr}^{\text{even}}W_i(0) = 0$

Proof. Since we take the even form trace, we can assume

$$W_i = cx_{i_1} \cdots x_{i_k} e_{j_1} \cdots e_{j_l} dy_{\beta_1} \cdots dy_{\beta_{2s}}.$$

If $k > 0$, we have $W_i(0) = 0$. When $k = 0$, $\chi(W_i) \leq 2(i-1) \leq 2([\frac{n}{2}] + 1 - 1) = n - 1$. By $\chi(dy_\beta) > 0$ and W_i being an odd element, we have $l < n$ is an odd integer, so $\text{tr}(e_{j_1} \cdots e_{j_l}) = 0$. \square

Lemma 2.3 Suppose $1 \leq j \leq [\frac{m}{2}] + 1$. If $W_{[\frac{n}{2}] + 1 + j}$ is an odd element and $\chi(W_{[\frac{n}{2}] + 1 + j}) \leq 2([\frac{n}{2}] + j)$, then

$$\phi_t \left(\frac{1}{t^{\frac{n}{2} + 1}} \text{Tr}^{\text{even}} W_{[\frac{n}{2}] + 1 + j}(0) t^{[\frac{n}{2}] + 1 + j} \right) = O(t^{\frac{1}{2}}), \quad t \searrow 0. \quad (2.21)$$

Proof. We can assume

$$W_{[\frac{n}{2}] + 1 + j} = c x_{i_1} \cdots x_{i_k} e_{j_1} \cdots e_{j_l} dy_{\beta_1} \cdots dy_{\beta_{2s}}.$$

Then $k = 0$ and $l = n$, otherwise $\text{Tr}^{\text{even}} W_{[\frac{n}{2}] + 1 + j}(0) = 0$. Since $\chi(W_{[\frac{n}{2}] + 1 + j}) \leq 2([\frac{n}{2}] + j) = n - 1 + 2j$, then $s \leq j - 1$ and $\phi_t(W_{[\frac{n}{2}] + 1 + j}) = t^{-s} W_{[\frac{n}{2}] + 1 + j}$. So the degree of t is $[\frac{n}{2}] + 1 + j - (\frac{n}{2} + s + 1) \geq \frac{1}{2}$. \square

By (2.20), Lemmas 2.2 and 2.3, we get

Theorem 2.4([BGS])

$$\text{Tr}^{\text{even}}[(D + \frac{c(T)}{4t})\exp(I^t)(x, x)] = O(t^{\frac{1}{2}}), \quad t \searrow 0. \quad (2.22)$$

3 The regularity of equivariant eta forms

Let isometry g act fibrewise on M and act as identity on B and g preserve the orientation and the spin structure on $S(TG)$. Let $\tilde{g} : \Gamma(S(TG)) \rightarrow \Gamma(S(TG))$ be the lift of g . We have the following definition of equivariant eta forms.

Definition 3.1 $\hat{\eta}_g$ denotes the even degree form on B

$$\hat{\eta}_g = \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} \text{Tr}^{\text{even}}[\tilde{g}(D + \frac{c(T)}{4t})\exp(I^t)] dt / t^{\frac{1}{2}}. \quad (3.1)$$

Since \tilde{g} is bounded operator, we can prove the regularity at $+\infty$ as in Section 2. Nextly, we prove the regularity at origin. Fix a $y \in B$ and consider the fibre G_y . Similar to Corollary 1.4 in [Z2], if g has no fixed points on G_y , then we can prove

$$\|\tilde{g}(D + \frac{c(T)}{4t})\exp(I^t)(x, gx)\| \leq \frac{C_1}{t^{\frac{n}{2} + \frac{m}{2} + 1}} \exp(-C_2/t), \quad (3.2)$$

where C_1, C_2 are constants and the norm represents the norm of coefficients of forms on B . So in this case, (3.1) is well defined. Since g is an isometry on G_y , for convenience, we assume the fixed point set F of g is connected and $\text{codim} F = 2n'$ and $\dim F = q$. Denote by $N(F)$ the normal bundle to F . Similar to the discussions of (3.2), we need only to prove that

$$\lim_{t \rightarrow 0} \left| \int_F \int_{N_\xi(\varepsilon)} \text{Tr}^{\text{even}}[\tilde{g}(D + \frac{c(T)}{4t})\exp(I^t)(x, gx)] dN_\xi d\xi \right| \leq C, \quad (3.3)$$

for some constant $C > 0$. Here $N_\xi(\varepsilon) = \{v \in N_\xi(F) \mid \|v\| < \varepsilon\}$. Similar to (2.18), we have

$$t dt \text{Tr}^{\text{even}}[\tilde{g}(D + \frac{c(T)}{4}) \exp(tI^1)(x, gx)] = \text{Tr}^{\text{even}}\{\tilde{g}h^{-1}(x) \exp[t(I^1 + dtu)(x, gx)]h(gx)\} \\ - \text{Tr}^{\text{even}}[\tilde{g} \exp(tI^1)(x, gx)]. \quad (3.4)$$

By (2.14), (2.16) and (3.4), we have

$$t\tilde{g}(D + \frac{c(T)}{4}) \exp(tI^1)(x, gx) \\ = \frac{e^{-d(x, gx)^2/4t}}{(4\pi t)^{\frac{n}{2}}} \tilde{g} \left\{ \sum_{i=1}^{2n'} (((dg - I)x)_{q+i} e_{q+i}) \left(\sum_{j=0}^{[\frac{n}{2}] + [\frac{m}{2}] + 2} U_j t^j + o(t^{[\frac{n}{2}] + [\frac{m}{2}] + 2}) \right) \right. \\ \left. + \sum_{i=0}^{[\frac{n}{2}] + [\frac{m}{2}] + 2} W_i t^i + o(t^{[\frac{n}{2}] + [\frac{m}{2}] + 2}) \right\}, \quad (3.5)$$

where $\chi(W_i) \leq 2(i-1)$ is an odd element. By [Z2, p.1126], we know that

$$\chi(\tilde{g}) \leq 2n', \quad \chi(\tilde{g} \sum_{i=1}^{2n'} (((dg - I)x)_{q+i} e_{q+i})) \leq 2n' - 2. \quad (3.6)$$

Lemma 3.2 Suppose $1 \leq j \leq [\frac{n}{2}] + [\frac{m}{2}] + 2$. If \overline{W} is an odd element and $\chi(\overline{W}) \leq 2n' + 2j - 2$, then

$$\lim_{t \rightarrow 0} \left(\frac{1}{t} \right)^{3/2} \left| \int_{N_\xi(\varepsilon)} \frac{e^{-d(x, gx)^2/4t}}{(4\pi t)^{\frac{n}{2}}} \phi_t[\text{tr}(\overline{W}(0; x))] t^j dx \right| \leq C_1, \quad (3.7)$$

for some constant $C_1 > 0$; where in the $W(z; x)$, z stands for tangential coordinates and x stands for normal coordinates.

Proof. We can assume that \overline{W} is a monomial, then it can be written as

$$\overline{W} = \phi(0) x_{i_1} \cdots x_{i_k} e_1 \cdots e_n dy_{\beta_1} \cdots dy_{\beta_{2s}}.$$

We note that we can assume that x_i in \overline{W} are normal coordinates, for otherwise $\text{tr} \overline{W}(0; \cdot) = 0$.

(i) If $\chi(\overline{W}) = 2n' + 2j - 2$, then $k = n + 2s - 2n' - 2j + 2$ is an odd integer. By making the change of variables $x = t^{1/2}b$, we get

$$\lim_{t \rightarrow 0} \left(\frac{1}{t} \right)^{3/2} \left| \int_{N_\xi(\varepsilon)} \frac{e^{-d(x, gx)^2/4t}}{(4\pi t)^{\frac{n}{2}}} \phi_t[\text{tr}(\overline{W}(0; x))] t^j dx \right| \\ \leq \lim_{t \rightarrow 0} \left(\frac{1}{t} \right)^{3/2} \left| \int_{N_\xi(\varepsilon/\sqrt{t})} \frac{e^{-\| (I - dg)b \|^2}}{(4\pi t)^{\frac{n}{2}}} t^{\frac{n}{2} + s - n' - j + 1} b_{i_1} \cdots b_{i_k} t^j \frac{1}{t^s} dy_{\beta_1} \cdots dy_{\beta_{2s}} t^{n'} db \right| = 0$$

(ii) If $\chi(\overline{W}) < 2n' + 2j - 2$, then $k > n + 2s - 2n' - 2j + 2$ and we get

$$\lim_{t \rightarrow 0} \left(\frac{1}{t} \right)^{3/2} \left| \int_{N_\xi(\varepsilon)} \frac{e^{-d(x, gx)^2/4t}}{(4\pi t)^{\frac{n}{2}}} \phi_t[\text{tr}(\overline{W}(0; x)) t^j dx] \right| \leq C_3.$$

We note that the above discussions are also correct when $n + 2s - 2n' - 2j + 2 < 0$. \square

By (3.5), (3.6) and Lemma 3.2, we get

Theorem 3.3

$$\text{Tr}^{\text{even}} \left[\tilde{g} \left(D + \frac{c(T)}{4t} \right) \exp(I^t)(x, gx) \right] = O(t^{\frac{1}{2}}), \quad t \searrow 0. \quad (3.8)$$

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References

- [APS]Atiyah, M. F.; Patodi, V. K.; Singer, I. M., Spectral asymmetry and Riemannian geometry. I. Math. Proc. Cambridge Philos. Soc. 77(1975), 43-69.
- [BGV]Berline, N.; Getzler, E.; Vergne, M., *Heat kernels and Dirac operators*. Springer-Verlag, Berlin, 1992.
- [BC1]Bismut, J. M.; Cheeger, J., Families index for manifolds with boundary, superconnections, and cones. I. Families of manifolds with boundary and Dirac operators. J. Funct. Anal. 89 (1990), no. 2, 313-363.
- [BC2]Bismut, J. M.; Cheeger, J., Families index for manifolds with boundary, superconnections and cones. II. The Chern character. J. Funct. Anal. 90 (1990), no. 2, 306-354.
- [BF]Bismut, J. M.; Freed, D., The analysis of elliptic families. II. Dirac operators, eta invariants, and the holonomy theorem. Comm. Math. Phys. 107 (1986), no. 1, 103-163.
- [BGS]Bismut, J.-M.; Gillet, H.; Soul, C. Analytic torsion and holomorphic determinant bundles. I. Bott-Chern forms and analytic torsion. Comm. Math. Phys. 115 (1988), no. 1, 49-78.
- [D]Donnelly, H., Eta invariants for G -spaces. Indiana Univ. Math. J. 27 (1978), no. 6, 889-918.
- [W]Wu, F., The Chern-Connes character for the Dirac operator on manifolds with boundary. *K-Theory* 7 (1993), no. 2, 145-174.
- [Z1]Zhang, W. P., Local Atiyah-Singer index theorem for families of Dirac operators. Differential geometry and topology, Lecture Notes in Math., 1369, Springer, Berlin, 1989 351-366.
- [Z2]Zhang, W. P., A note on equivariant eta invariants. Proc. Amer. Math. Soc. 108 (1990), no. 4, 1121-1129.

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